

On the computational complexity of degenerate unit-distance representations of graphs

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Abstract

Some graphs admit drawings in the Euclidean k -space in such a (natural) way, that edges are represented as line segments of unit length. Such drawings will be called k dimensional unit distance representations. When two non-adjacent vertices are drawn in the same point, we say that the representation is degenerate. The dimension (the Euclidean dimension) of a graph is defined to be the minimum integer k needed that a given graph has non-degenerate k dimensional unit distance representation (with the property that non-adjacent vertices are mapped to points, that are not distance one appart).

It is proved that deciding if an input graph is homomorphic to a graph with dimension $k \geq 2$ (with the Euclidean dimension $k \geq 2$) are NP-hard problems.

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1 Introduction and background

Define a *representation* $\rho := (\rho_V, \rho_E)$ of a graph G in a set M as a mapping $\rho_V : V(G) \rightarrow M$ and a mapping $\rho_E : E(G) \rightarrow 2^M$, such that if v is an end-vertex of an edge $e = u \sim v$, then $\rho_V(v) \in \rho_E(e)$ (and $\rho_V(u) \in \rho_E(e)$) [7]. If the converse is also true, that is, that for $v \in V(G)$ and $e \in E(G)$, if $\rho_V(v) \in \rho_E(e)$ is true, then v is an end-vertex of $e = u \sim v$, the representation is called a *realization*. If there is no danger of confusion we drop the subscripts and denote both mappings ρ_V and ρ_E by ρ . Observe, that the intersection $\rho(e) \cap \rho(\ell)$ of realizations of non-adjacent edges $e, \ell \in E(G)$ can be non-empty, but cannot contain a realization of any vertex of G . Thus, representations of edges of G may cross, which is different from the usual graph embedding case. If the mapping ρ is not injective on $V(G)$ it is called *degenerate*.

In this paper we consider only representations in the Euclidean k -space, i.e. (k dimensional Euclidean) representations that have $M = \mathbb{R}^k$, and where an edge is always represented by the line segment between the representations of its end-vertices. Each Euclidean realization of a connected graph on at least three vertices is non-degenerate, see [11]. An Euclidean representation is called a *unit distance representation* if each edge is represented as a line segment of length one. A graph is called a k dimensional *unit distance graph* if it has a k dimensional unit distance realization. For example, the well known Petersen graph $G(5, 2)$ is a unit distance graph with exactly 18 different degenerate unit distance representations, see [10].

Erdős *et al.*, see [6], defined the *dimension* of a graph G , denoted as $\dim(G)$, as the minimum integer k needed that G is a k dimensional unit distance graph. Other authors (e.g. [2, 14, 15]) defined an *unit distance embedding* to be a unit distance realization that maps non-adjacent vertices to points with distances other than one. The smallest integer k needed that a given graph G has an unit distance embedding in \mathbb{R}^k is called the *Euclidean dimension* of G , see for example [13], and is, as usual, denoted as $e(G)$. A graph with $e(G) = k$ is called a k dimensional *strict unit distance graph* (or, following Boben *et al.* [1], k dimensional *unit distance coordinatization*). As any k dimensional strict unit distance graph is a k dimensional unit distance graph, it follows that for a given graph G , $\dim(G) \leq e(G)$. Wheh $\dim(G) = 2$ (or $e(G) = 2$) we will be talking about *planar* unit distance graphs (or planar strict unit distance graphs). A planar graph with a planar unit distance realization in the plane is called a *matchstick* graph.

If G is composed of the connected components G_1, G_2, \dots, G_s , then G admits a unit distance representation in \mathbb{R}^k if and only if each component G_i , $1 \leq i \leq s$, admits a unit distance representation in \mathbb{R}^k . By definition, loops are not valid unit distance Euclidean edge representations, and there exists only one line segment of length one between two points in the Euclidean space. Therefore, we may restrict our attention to connected simple graphs.

A mapping $f : V(G) \rightarrow V(H)$ from a graph G into a graph H is called a *graph homomorphism*, if f maps vertices of G into vertices of H , such that $u \sim v \in E(G)$ implies $f(u) \sim f(v) \in E(H)$ (f preserves adjacencies). Each

graph morphism $f : V(G) \rightarrow V(H)$ induces a unique mapping (denoted by the same letter) $f : E(G) \rightarrow E(H)$, such that for $e = u \sim v$, $f(e) = f(u) \sim f(v)$ (see e.g. [8]). Let G and H be graphs, let $f : G \rightarrow H$ be a graph homomorphism, and let $k \in \mathbb{N}$. Any unit distance representation of H in \mathbb{R}^k (if it exists) can be extended to a (possibly degenerate) unit distance representation of G in \mathbb{R}^k , see [11].

The union $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The disjoint union of two graphs G_1 and G_2 , denoted $G_1 \tilde{\cup} G_2$, is the graph obtained by taking the union of G_1 and G_2 on disjoint vertex sets, $V(G_1)$ and $V(G_2)$. The join of two simple graphs G_1 and G_2 , written $G_1 + G_2$ is the graph obtained by taking the disjoint union of G_1 and G_2 and adding all edges joining $V(G_1)$ and $V(G_2)$.

A wheel graph W_n on $n \geq 4$ vertices is defined as $W_n = W_{1,n-1} := K_1 + C_{n-1}$, where K_1 is the one vertex graf and C_{n-1} the cycle graph on $n-1$ vertices. It is well known [2] that $\dim(W_n) = 3$ for $n \neq 7$ and $\dim(W_7) = 2$. Even though $e(W_5) = \dim(W_5) = 3$, there exists degenerate planar unit distance representations of the graph W_5 ; see Figure 1.

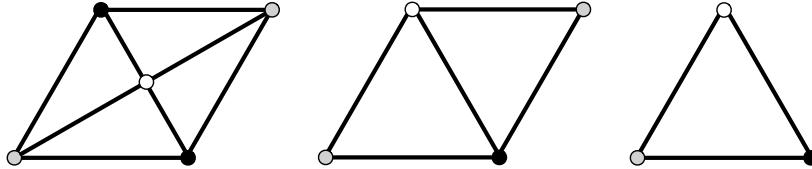


Figure 1: The wheel graph W_5 (on the left), which admits a proper 3 coloring, is not a planar unit distance graph. Degenerate planar unit distance representations of the wheel graph W_5 are obtained using black vertex identification (in the middle) and additional gray vertex identification (on the right).

It is therefore interesting to answer a question “Does, for a given graph G and a natural number k , there exists a degenerate k dimensional unit distance representation of G ?”. One can instead, answer a question “For a given natural number k and a graph G , is G homomorphic to a graph with dimension k ?”. Of course, similar questions concerning the Euclidean dimension can be asked.

The only graph with zero dimension is K_1 , and the same holds for the Euclidean dimension. It can be easily shown that the only graphs with dimension one are path graphs on at least two vertices. Again, the same is true for the Euclidean dimension. Thus, we can check in polynomial time whether a given graph is homomorphic to a linear strict unit distance graph. In Section 2 we prove that deciding if an input graph is homomorphic to a planar unit distance graph (planar strict unit distance graph) are NP-hard problems. In Section 3 we consider the same problems in higher dimensions.

2 Planar unit distance representations

Theorem 1 *Deciding if an input graph G is homomorphic to a graph H with $\dim(H) = 2$ (with $e(H) = 2$) are NP-hard problems.*

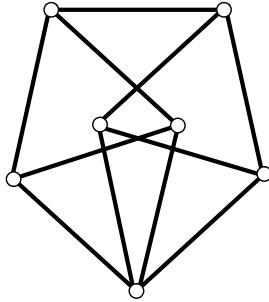


Figure 2: The well known Moser graph (Moser spindle) is a planar strict unit distance graph.

Proof The proof is based on the Moser graph (see Fig. 2) which is a well known planar strict unit distance graph. It has the property that every homomorphism to a unit distance graph H is injective and the image H is isomorphic to the Moser graph itself. In particular, adding any edge to the Moser graph results in a graph with dimension at least 3. It is not hard to see, that the dimension and the Euclidean dimension of the Moser graph are both equal to two. Using Laman's Theorem, see e.g. [9], it is not hard to show that the unit distance coordinatization of Moser graph is rigid.

The NP-hardness reduction goes from 3 – SATISFIABILITY. Given a formula Φ with a set of variables X and a set of clauses C (each clause containing exactly 3 literals), we construct a graph G_Φ , such that G_Φ is homomomorphic to a strict unit distance graph (namely to the Moser graph) if Φ is satisfiable, while G_Φ is not homomorphic to any unit distance graph if Φ is unsatisfiable. This will prove both claims.

The cornerstone of the construction is a copy of the Moser graph with vertices $\{u, T, F, u', u'', v, w\}$ and edges as drawn in the shaded part of the Fig. 3. For every variable $x \in X$, we add an edge $x' \sim x''$, where x' represents the positive literal and x'' the negation of x . Also each x' and x'' are made adjacent to both u and u' . The part of the construction of G_Φ that corresponds to a variable $x \in X$ can be seen in the bolded part of the Fig. 3.

For every clause $c \in C$, say $c = (\lambda_{c,1} \vee \lambda_{c,2} \vee \lambda_{c,3})$, the clause gadget is depicted in Fig. 4. It contains six vertices, three of them being connected to the literals appearing in c (one to each) and one adjacent to the special vertex F of the Moser graph.

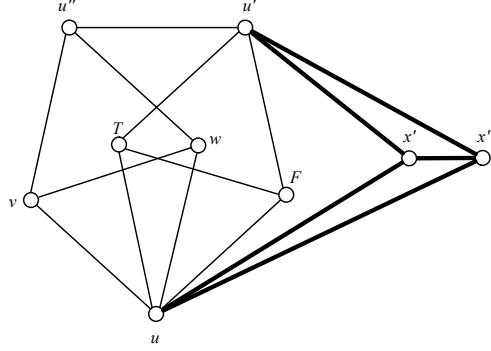


Figure 3: The variable part of the construction of G_Φ .

Formally,

$$V(G_\Phi) = \{u, v, w, u', u'', T, F\} \cup \bigcup_{x \in X} \{x', x''\} \cup \bigcup_{c \in C} \{c_1, c_2, c_3, c_4, c_{12}, c_{34}\}$$

and

$$\begin{aligned} E(G_\Phi) = & \{u \sim T, u \sim F, T \sim F, T \sim u', F \sim u', u \sim v, u \sim w, v \sim w, \\ & v \sim u'', w \sim u'', u' \sim u''\} \cup \\ & \bigcup_{x \in X} \{x' \sim x'', x' \sim u, x'' \sim u, x' \sim u', x'' \sim u'\} \cup \\ & \bigcup_{c \in C} \{c_1 \sim c_2, c_2 \sim c_3, c_3 \sim c_4, c_4 \sim c_1, c_1 \sim c_{12}, c_2 \sim c_{12}, c_3 \sim c_{34}, \\ & c_4 \sim c_{34}, c_{12} \sim F, c_2 \sim \lambda_{c,1}, c_3 \sim \lambda_{c,2}, c_{34} \sim \lambda_{c,3}\}. \end{aligned}$$

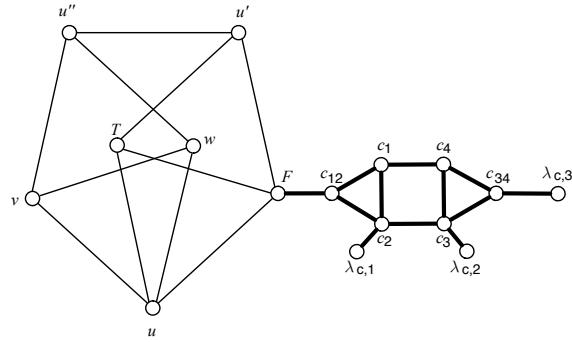


Figure 4: The clause part of G_Φ .

Suppose first that G_Φ is homomorphic to a graph H with $\dim(H) = 2$ (to a planar unit distance graph H). Consider a planar unit distance realization of H . There are at most two paths of length two between two vertices of a planar unit distance graph [14]. Since there are four paths of length two between u and u' in G_Φ , the homomorphism either places x' in T and x'' in F , or vice versa. This placement defines a truth assignment on the variables of Φ - we say that x is **true** if x' is placed in T , and that it is **false** otherwise. We claim that Φ is satisfied by this assignment. Suppose there is an unsatisfied clause, say $c = (\lambda_{c,1} \vee \lambda_{c,2} \vee \lambda_{c,3})$. Then all three vertices $\lambda_{c,1}, \lambda_{c,2}, \lambda_{c,3}$ must be placed in F , and the clause gadget must map onto another copy of the Moser graph (with F being its degree four vertex). But then, in the plane, the edge $c_2 \sim c_3$ cannot have a unit length. We have a contradiction.

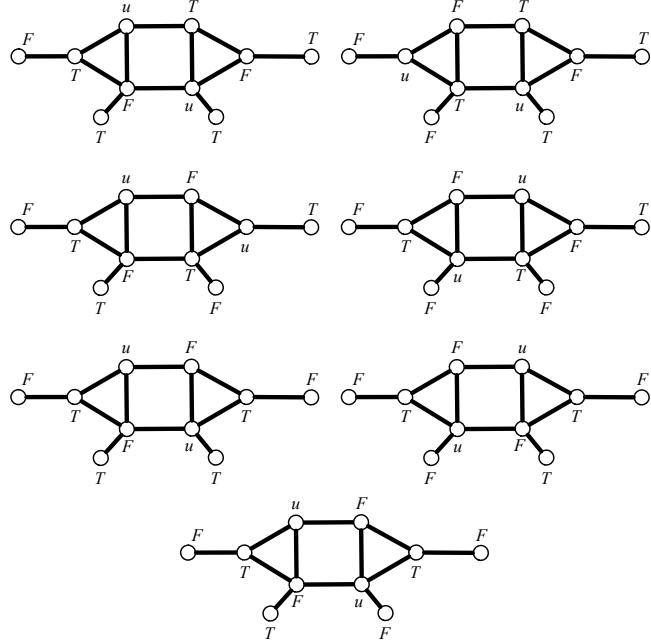


Figure 5: Case analysis of placement of the vertices of clause gadgets.

Suppose now that Φ is satisfied by a truth assignment $\phi : X \rightarrow \{\text{true}, \text{false}\}$. We fix a strict planar unit distance realization of the Moser graph, place x' in T and x'' in F whenever $\phi(x) = \text{true}$ (and vice versa if $\phi(x) = \text{false}$). Following the case analysis in Fig. 5, the vertices of the clause gadgets can be placed in vertices u, T and F . Hence, we have constructed a well defined surjective homomorphism of G_Φ onto the Moser graph. Therefore, G_Φ is homomorphic to a graph with the Euclidean dimension 2. qed

3 k dimensional unit distance representations

Let $S_k(\vec{s}, r)$ denote the k dimensional (hyper)sphere in \mathbb{R}^k with center in \vec{s} and radius r . When the center and the radius of a sphere are not important, the abbreviation S_k will be used.

Lemma 1 *Let $k > 1$ be a natural number. A non-empty non-degenerated intersection of two k dimensional spheres with distinct centers is $k - 1$ dimensional sphere.*

Proof Let $c, r, R > 0$ be positive real numbers. We can, without loss of generality, assume, that the first sphere is centered at the origin $\vec{0}$ and the second one is centered on the x_1 axes in $\vec{c} = (c, 0, \dots, 0)$. Spheres $S_k(\vec{0}, r)$ and $S_k(\vec{c}, R)$ are determined by equations

$$x_1^2 + x_2^2 + \dots + x_k^2 = r^2$$

and

$$(x_1 - c)^2 + x_2^2 + \dots + x_k^2 = R^2.$$

We subtract the above equations to obtain a linear equation for x_1 with a solution $a := \frac{r^2 + c^2 - R^2}{2c}$. When a is inserted into the first equation, the equality $x_2^2 + \dots + x_k^2 = r^2 - a^2$ is obtained. When $r^2 - a^2 < 0$, the system does not have any solution and this case corresponds to a situation, where spheres do not intersect. A situation $|a| = r$ corresponds to spheres that are tangent to each other and their intersection is a point - a degenerated sphere. Assume now, that $r^2 - a^2 > 0$. The equation $x_2^2 + \dots + x_k^2 = r^2 - a^2$ determines the $k - 1$ dimensional sphere in the hyperplane $x_1 = a$ in \mathbb{R}^k . qed

Lemma 2 *Let S_2 denote the circle in the Euclidean plane, which is circumscribed to a unit distance realization of the complete graph K_3 on three vertices. Let G be a connected graph with (possibly degenerate) planar unit distance representation, which places all vertices of G into points that lay on S_2 . Then $\chi(G) \leq 3$.*

Proof Graph K_3 has unique (unique up to isometries) unit distance realization in the plane. It can be represented as an equilateral triangle with all sides of unit length, which has unique circumscribed circle. Denote the circumscribed circle by S_2 . Every point on S_2 has exactly two points that are unit distance apart from it and belong to S_2 . Graph G has a unit distance representation on S_2 and can hence occupy only three distinct points on S_2 , which are exactly the vertices of (possibly another) equilateral triangle with all sides of length one. Thus, there exists a proper 3 coloring of G - colors being the vertices of the equilateral triangle. qed

Lemma 2 can be generalized. Let $G_{k,r,\alpha}$ denote the graph with vertices being points of k dimensional sphere $S_k(\vec{0}, r)$ with radius r in \mathbb{R}^k , where two vertices are connected if and only if they are at distance α . L. Lovász [12] proved the following inequalities.

Theorem 2 Let $k \geq 3$ be a natural number. Then, for $0 \leq \alpha \leq 2$ holds

$$k \leq \chi(G_{k,1,\alpha})$$

and

$$\chi\left(G_{k,1,\sqrt{\frac{2(k+1)}{k}}}\right) \leq k+1.$$

Corollary 1 Let $k \geq 3$ be a natural number and let S_k denote the k dimensional sphere in \mathbb{R}^k , which is circumscribed to the unit distance realization of the complete graph K_{k+1} on $k+1$ vertices. Let G be a connected graph with (possibly degenerate) k dimensional unit distance representation, which places all vertices of G into points that lay on S_k . Then $\chi(G) \leq k+1$.

Proof It is known, that the circumradius of the (hyper)sphere that is circumscribed to the regular simplex with $k+1$ vertices and all sides of length ℓ , equals to $\ell\sqrt{\frac{k}{2(k+1)}}$. Thus, the circumradius of sphere S_k equals to $\sqrt{\frac{k}{2(k+1)}}$. The representation of graph $G_{k,\sqrt{\frac{k}{2(k+1)}},1}$ on sphere S_k can be (down)scaled to obtain the representation on k dimensional unit sphere (with radius one), that is circumscribed to the regular simplex with all sides of length $\sqrt{\frac{2(k+1)}{k}}$. Following Theorem 2, $\chi\left(G_{k,1,\sqrt{\frac{2(k+1)}{k}}}\right) \leq k+1$. The proper $k+1$ coloring of $G_{k,1,\sqrt{\frac{2(k+1)}{k}}}$ gives rise to the proper $k+1$ coloring of $G_{k,\sqrt{\frac{k}{2(k+1)}},1}$. Since G is a subgraph of $G_{k,\sqrt{\frac{k}{2(k+1)}},1}$, $\chi(G) \leq k+1$. qed

Theorem 3 Let $k \geq 3$ be a natural number. Deciding if an input graph G is homomorphic to a graph H with $\dim(H) = k$ (with $e(H) = k$) are NP-hard problems.

Proof Let K'_k and K''_k be two copies of the complete graph with $k \geq 3$ vertices. Let v, w', w'' be additional vertices, such that $v, w', w'' \notin V(K'_k) \cup V(K''_k)$. Denote $M'_k = K'_k + \{v, w'\}$, $M''_k = K''_k + \{v, w''\}$ and $M_k = M'_k \cup M''_k \cup \{w' \sim w''\}$, e.g. Figure 6. Graph M_k is well known Moser-Raiskii spindle.

Note that $e(K'_k) = \dim(K'_k) = k-1$. Since $K'_k + v$ is the complete graph on $k+1$ vertices, $\dim(K'_k + v) = e(K'_k + v) = k$. Applying another graph join does not change any of dimensions. Thus, $e(M'_k) = e(M''_k) = \dim(M'_k) = \dim(M''_k) = k$. Both M'_k and M''_k are rigid in \mathbb{R}^k . Consider a unit distance realization of $M'_k \cup M''_k$ in \mathbb{R}^k . We can rotate M''_k around the representation of the vertex v , such that w' and w'' results distance one apart. Hence $\dim(M_k) = k$. Additionally, we can rotate the subgraph K''_k in such a way, that $e(M_k) = k$. Note that H_k has the property that every k dimensional unit distance realization in \mathbb{R}^k , places vertices v and w' in distinct points (this is guaranteed by the edge $w' \sim w''$).

Let us construct the graph H_k , such that $V(H_k) := V(G) \cup V(M_k)$ and

$$E(H_k) := E(G) \cup E(M_k) \cup \bigcup_{u \in V(G)} \{u \sim v, u \sim w'\},$$

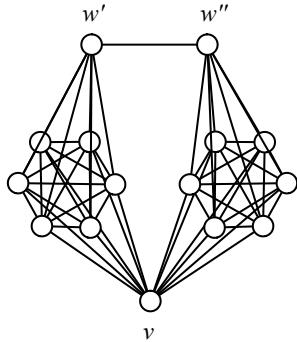


Figure 6: The rigid graph M_6 contains two similar subgraphs, which are obtained as a graph join of the complete graph on six vertices and two disconnected vertices.

where vertices v, w' are vertices from M_k .

We reduce from graph 3 – COLORABILITY. We will prove that, when H_k is homomorphic to a graph with the dimension k , then $\chi(G) \leq k$, while from $\chi(G) \leq k$ follows, that H_k is homomorphic to a graph with the Euclidean dimension k .

Suppose H_k is homomorphic to a graph with the dimension k . Then H_k has (possibly degenerate) unit distance representation ρ in \mathbb{R}^k . All vertices of G are one apart from $\rho(v)$ and from $\rho(w')$, and they all belong to the intersection of two unit k dimensional spheres with centers in $\rho(v)$ and $\rho(w')$. Using Lemma 1, a non-empty non-degenerated intersection of two k dimensional spheres with distinct centers is $k - 1$ dimensional sphere. All vertices of the complete graph K'_k belong to the same intersection and hence to the same $k - 1$ dimensional sphere. If G is connected, we can use Corollary 1 and $\chi(G) \leq k$. If G is not connected, we can observe its components in similar matter.

Assume now, that $\chi(G) \leq k$. Using proper k -coloring of G we can map vertices of G into vertices of K'_k , see [8]. Hence a homomorphic image of H_k is M_k and $e(M_k) = k$.

As any k dimensional strict unit distance graph is a k dimensional unit distance graph, it follows that for a given graph G , $\dim(G) \leq e(G)$. Hence we proved both claims of this Theorem. qed

4 Concluding remarks

There are several results in connection to computational complexity and unit distance graphs. Firstly, it is known, that a graph reconstruction problem is strongly NP-complete in 1 dimension and strongly NP-hard in higher dimensions, see [16]. For general edge-lengths, graph realization problems and prob-

lems concerning rigidity of structures in connection to computational complexity, were observed in [9] and [17].

P. Eades and S. Whitesides proved in [4] that the decision problem: “Is a planar graph G a matchstick graph?” is NP-hard. They used a reduction from the well known NP-complete problem NOT ALL EQUAL 3 – SATISFIABILITY, a special instance of the 3 – SATISFIABILITY problem, in which, each term has to have at least one true and at least one false literal. Their proof was constructive, they used a matchstick graph to construct a mechanical device, called *logic engine*, that was used to simulate the NOT ALL EQUAL 3 – SATISFIABILITY problem instance. Cabello *et al.* in [3] used similar approach to prove that the decision problem: “Is a planar 3-connected infinitesimally rigid graph G a matchstick graph?” is NP-hard.

Additionally, P. Eades and N. C. Wormald proved in [5] that the decision problem: “Is a 2-connected graph G a matchstick graph?” is NP-hard, too.

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